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ON GRUNDY NUMBER OF LADDER GRAPH FAMILIES

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ABSTRACT:
The Grundy number of a graph G, denoted by $\Gamma(G)$, is the maximum number required for proper grundy coloring. This Grundy Coloring (also known as First-Fit Coloring) is defined as $f:V(G)\to\{C_t : t\in \mathrm{N}\}$ such that $\forall$ $f(m)=C_t$, m is adjacent with all C_{t-1} colors where $m\in V(G)$. In this, we obtained the grundy number of some graphs from ladder graph family such as Ladder graph $[L_n]$, Open Ladder graph $[O(L_n)]$, Slanting Ladder graph $[S(L_n)]$, Triangular Ladder graph $[T(L_n)]$, Open Triangular Ladder graph $[O(TL_n)]$, Circular Ladder graph $[C(L_n)]$, Mobius Ladder graph $[M_n]$, Diagonal Ladder graph $[D(L_n)]$, Open Diagonal Ladder graph $[O(DL_n)]$.
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1. INTRODUCTION

In this paper, the graphs $G=V(G),E(G)$ we use are simple, finite, connected and undirected graphs where $V(G)$ is the set of vertices in G & $E(G)$ is the set of edges in G. A Proper k-coloring is the assignment of colors to $V(G)$ in G such the no two adjacent vertices bear same color. This proper k-coloring is said to be grundy k-coloring if it satisfies the condition: $f(u)=C_k$ then $u\sim C_1, u\sim C_2, u\sim C_3,\ldots,u\sim C_{k-1}$ for the mapping $f:V(G)\to\{C_k : k\in N\}$. In other words, a vertex colored with C_k must be adjacent with all C_{k-1} colors[2]. This grundy coloring is also known as first-fit coloring[7]. It is a maximum coloring which was initially initiated by P M Grundy for directed version but the undirected version was introduced by Christen and Selkow[5][8]. The grundy number $\Gamma(G)$ can also be predicted by using greedy algorithm which consider $V(G)$ in some order and colors them with proper first available color. It is well known that $\omega(G)\le\chi(G)\le\Gamma(G)\le\Delta(G)+1$ where $\omega(G)$ is the clique number[6]. In developing the concept of grundy coloring, [1]Brice Effatin found some exact values for the grundy coloring of some central graphs. In a manuscript [11] the upper bound for grundy number of a graph in terms of its Randic index, order and clique number were discussed.

2. PRELIMINARIES

Definition 2.1: A Grundy n-coloring of G is an n-coloring of G such that $\forall$ color C_t, every node colored with C_t is adjacent to atleast one node colored with C_s $\forall$ $C_s < C_t$. The Grundy number $\Gamma(G)$ is the maximum number n such that G is Grundy n-coloring[3].

Definition 2.2: A Ladder graph L_n is defined by $L_n = P_n\times K_2$ where P_n is the path with n vertices, K_2 is the complete graph with two vertices and $\times$ denotes the cartesian product[10].

Definition 2.3: A Open Ladder graph $O(L_n)$ where $n\ge 2$ is obtained from two paths of length $n-1$ with $V(G)=\{u_i : 1\le i\le n\}\cup\{v_j : 1\le j\le n\}$ and $E(G)=\{u_iu_{i+1} : 1\le i\le n-1\}\cup\{v_jv_{j+1} : 1\le j\le n-1\}\cup\{u_iv_j : i,j\in[2,n-1]\}$ [10].

Definition 2.4: A Slanting Ladder graph $S(L_n)$ is the graph obtained from two paths $\{u_i : 1 \le i \le n\}$ & $\{v_j : 1 \le j \le n\}$ by joining each u_i with v_{i+1} $\forall$ $1 \le i \le n-1$[10].

Definition 2.5: A Triangular Ladder graph $T(L_n)$ where $n \ge 2$ is a graph obtained from L_n by adding the edges $u_i v_{i+1} : 1 \le i \le n-1$. The vertices of L_n are u_i and v_i (u_i and v_i are the two paths in L_n)[10].

Definition 2.6: A Open Triangular Ladder graph $OT(L_n)$, where $n \ge 2$, is obtained from an open ladder by adding the edges $\{u_i v_{i+1} : 1 \le i \le n-1\}$[10].

Definition 2.7: A Circular Ladder graph $C(L_n)$ is the union of an outer cycle $C_0 : u_1 u_2 u_3 ... u_n u_1$ and an inner cycle $C_1 : v_1 v_2 v_3 ... v_n v_1$ with additional edges $(u_i v_i)$, $i = 1, 2, 3, ..., n$ called spokes[10].

Definition 2.8: A Mobious Ladder graph M_n is the graph obtained from the ladder graph by joining the opposite end points of the two copies of P_n[10].

Definition 2.9: A Diagonal Ladder graph $D(L_n)$, $n \le 2$ is a ladder graph with 2n vertices & is constructed from a ladder graph L_n by adding the edges $u_i v_{i+1}$ and $u_{i+1} v_i$ for $1 \le i \le n-1$[9].

Definition 2.10: A Open Diagonal Ladder graph $OD(L_n)$ is obtained from a diagonal ladder graph by removing the edges u_i & v_i for $i = 1, n$[10].

3. MAIN RESULTS

Here, we concentrate on the grundy number of ladder graph families such as Ladder, Open Ladder, Slanting Ladder, Triangular Ladder, Open Triangular Ladder, Circular Ladder, Mobius Ladder, Diagonal Ladder & Open Diagonal Ladder which are denoted by L_n, $O(L_n)$, $S(L_n)$, $T(L_n)$, $O(TL_n)$, $C(L_n)$, M_n, $D(L_n)$ and $O(DL_n)$ respectively.

Theorem 3.1: For $n \ge 1$, the grundy number for ladder graph L_n is given by

$$\Gamma(L_n) = \begin{cases} 2, & n = 1, 2 \\ 4, & n \ge 3 \end{cases}$$

Proof: Consider a ladder graph L_n with vertex set $V(L_n) = \{u_i : 1 \le i \le n\} \cup \{v_j : 1 \le j \le n\}$ and edge set $E(L_n) = \{u_i u_{i+1} : 1 \le i < n\} \cup \{v_j v_{j+1} : 1 \le j < n\} \cup \{u_i v_j : i, j \in [1, n] \,\&\, i = j\}$ with $|V(L_n)| = 2n$ and $|E(L_n)| = 3n - 2$.

$\therefore$ We have $\Delta(L_n) = \begin{cases} n, & n = 1, 2 \\ 3, & n \ge 3 \end{cases}$ and $\delta(L_n) = \begin{cases} n, & n = 1 \\ 2, & n \ge 2 \end{cases}$

Consider the colors $C_1, C_2, C_3, ...$ and assign the colors as follows.

Case 1: For $n = 1, 2$

Let us consider the mapping $\Pi : V(L_n) \to \{C_s : 1 \le s \le 2\}$ and assign the colors as follows.

Subcase 1: When $n = 1$

- $\Pi(u_n) = C_2$
- $\Pi(v_n) = C_1$

Obviously, $\Gamma(L_n) = 2$ since $\Delta(L_n) = n$.

Subcase 2: When $n = 2$

- $\Pi(u_n) = \Pi(v_{n-1}) = C_2$

- $\Pi(v_n) = \Pi(u_{n-1}) = C_1$

Thus, $\Gamma(L_n) = 2$. Suppose $\Gamma(L_n) = 3$, then the vertex u_{n-1} colored with C_2 is not adjacent with C_1 for the mapping $\Pi(u_n) = \Pi(v_{n-1}) = C_3$, $\Pi(u_{n-1}) = C_2$ and $\Pi(v_n) = C_1$ which contradicts the definition of grundy coloring.

$\therefore$ From the above subcases, we have $\Gamma(L_n) = 2$ for $n = 1, 2$.

Case 2: For $n \geq 3$

Assign the colors by using the mapping $\phi : V(L_n) \to \{C_t : 1 \leq t \leq 4\}$

- For $i = j = 2$, $\phi(u_i) = C_4$, $\phi(v_j) = C_3$, $\phi(u_{i-1}) = \phi(v_{j+1}) = C_2$ and $\phi(u_{i+1}) = \phi(v_{j-1}) = C_1$
- For $i, j \in [4, n]$, $\phi(u_i) = \begin{cases} C_2, & i \equiv 0 \bmod 2 \\ C_1, & i \equiv 1 \bmod 2 \end{cases}$ and $\phi(v_j) = \begin{cases} C_2, & j \equiv 1 \bmod 2 \\ C_1, & j \equiv 0 \bmod 2 \end{cases}$

Obviously, $\Gamma(L_n) = 4$ for $n \geq 3$. Suppose $\Gamma(L_n) < 4$, eventhough it satisfies the definition of grundy coloring, it is not maximum.

Thus, from the above cases, we have $\Gamma(L_n) = \begin{cases} 2, & n = 1, 2 \\ 4, & n \geq 3 \end{cases}$

Theorem 3.2: For $n \geq 3$, the grundy number for open ladder graph $O(L_n)$ is given $\Gamma[O(L_n)] = \begin{cases} 3, & n = 3, 4 \\ 4, & n \geq 5 \end{cases}$

Proof: Consider a open ladder graph $O(L_n)$ with vertex set $V[O(L_n)] = \{u_i : 1 \leq i \leq n\} \cup \{v_j : 1 \leq j \leq n\}$ and edge set $E[O(L_n)] = \{u_i u_{i+1} : 1 \leq i < n\} \cup \{v_j v_{j+1} : 1 \leq j < n\} \cup \{u_i v_j : i, j \in (1, n) \,\&\, i = j\}$ with $|V[O(L_n)]| = 2n$ and $|E[O(L_n)]| = 3n - 4$.

$\therefore$ We have $\Delta[O(L_n)] = 3$ and $\delta[O(L_n)] = 1$.

Consider the colors $C_1, C_2, C_3, \ldots$ and assign the colors as follows.

Case 1: When $n = 3, 4$

Let us consider the mapping $\alpha : V[O(L_n)] \to \{C_k : 1 \leq k \leq 3\}$ and assign the colors as follows.

- $\alpha(u_1) = \alpha(v_1) = \alpha(u_n) = \alpha(v_n) = C_1$
- $\alpha(u_{n-1}) = C_3$ & $\alpha(v_{n-1}) = C_2$
- For $n = 4$, $\alpha(u_{n-2}) = C_2$ & $\alpha(v_{n-2}) = C_3$

Thus, $\Gamma[O(L_n)] = 3$. Suppose $\Gamma[O(L_3)] = 4$, then the vertex u_1 & v_n colored with C_2 is not adjacent with C_1 for the mapping $\alpha(u_{n-1}) = C_4$, $\alpha(v_{n-1}) = C_3$, $\alpha(u_1) = \alpha(v_n) = C_2$ & $\alpha(u_n) = \alpha(v_1) = C_1$ which contradicts the definition of grundy coloring. Similarly $\Gamma[O(L_4)] = 4$ also contradicts grundy coloring. And $\Gamma[O(L_n)] > 4$ is not possible since $\Gamma \leq \Delta + 1$.

$\therefore$ $\Gamma[O(L_n)] = 3$ for $n = 3, 4$.

Case 2: When $n \geq 5$

Let us consider the mapping $\beta : V[O(L_n)] \to \{C_l : 1 \leq l \leq 4\}$ such that

- For $i, j = 3$, $\beta(u_i) = C_4$, $\beta(v_j) = C_3$, $\beta(u_{i-1}) = C_2$ & $\beta(v_{j-1}) = C_1$

- $\beta(u_1)=C_1$ & $\beta(v_1)=C_2$
- For $i,j\in[4,n]$, $\beta(u_i)=\begin{cases}C_2, & i\equiv 1 \bmod 2\\ C_1, & i\equiv 0 \bmod 2\end{cases}$ and $\beta(v_j)=\begin{cases}C_2, & j\equiv 0 \bmod 2\\ C_1, & j\equiv 1 \bmod 2\end{cases}$

Obviously, $\Gamma[O(L_n)]=4$ for $n\ge 5$. Suppose $\Gamma[O(L_n)]<4$, eventhough it satisfies the definition of grundy coloring, it is not maximum.

Thus, from the above cases, we have $\Gamma[O(L_n)]=\begin{cases}3, & n=3,4\\ 4, & n\ge 5\end{cases}$

Theorem 3.3: For $n\ge 2$, the grundy number for slanting ladder graph $S(L_n)$ is given by

$$\Gamma[S(L_n)]=\begin{cases}3, & n=2,3\\ 4, & n\ge 4\end{cases}$$

Proof: Consider a slanting ladder graph $S(L_n)$ with vertex set $V[S(L_n)]=\{u_i:1\le i\le n\}\cup\{v_j:1\le j\le n\}$ and edge set $E[S(L_n)]=\{u_iu_{i+1}:1\le i<n\}\cup\{v_jv_{j+1}:1\le j<n\}\cup\{u_iv_{i+1}:1\le i<n\}$ with $|V[S(L_n)]|=2n$ and $|E[S(L_n)]|=3n-3$.

$\therefore$ We have $\Delta[S(L_n)]=\begin{cases}2, & n=2\\ 3, & n\ge 3\end{cases}$ and $\delta[S(L_n)]=1$

Consider the colors $C_1,C_2,C_3,\dots$ and assign the colors as follows.

Case 1: When $n=2,3$

Consider the mapping $\rho:V[S(L_n)]\to\{C_s:1\le s\le 3\}$ and assign the colors as follows.

- $\rho(u_n)=\rho(v_{n-1})=C_1$
- $\rho(v_n)=C_2$
- $\rho(u_{n-1})=C_3$
- For $n=3$, $\rho(u_1)=\rho(v_1)=C_2$

Thus, $\Gamma[S(L_n)]=3$. Obviously $\Gamma[S(L_n)]=3$. Suppose $\Gamma[S(L_3)]>3$, then the vertex v_n colored with C_3 is not adjacent with C_2 for the mapping $\rho(u_1)=\rho(v_1)=C_2$, $\rho(u_n)=\rho(v_{n-1})=C_1$, $\rho(u_{n-1})=C_4$ and $\rho(v_n)=C_3$ which contradicts the definition of grundy coloring.

$\therefore$ $\Gamma[S(L_n)]=3$ for $n=2,3$.

Case 2: When $n\ge 4$

Let us consider the mapping $\gamma:V[S(L_n)]\to\{C_t:1\le t\le 4\}$ such that

- $\gamma(u_1)=\gamma(v_1)=C_1$
- $\gamma(u_i)=\begin{cases}C_4, & i=2\\ C_2, & i=3\end{cases}$
- $\gamma(v_j)=C_j \;\forall\; j=2,3$
- $\forall\; i,j\in[4,n]$, $\gamma(u_i)=\gamma(v_j)=\begin{cases}C_2, & i,j\equiv 1 \bmod 2\\ C_1, & i,j\equiv 0 \bmod 2\end{cases}$

Obviously, $\Gamma[S(L_n)]=4$ for $n \geq 4$. Suppose $\Gamma[S(L_n)]<4$, eventhough it satisfies the definition of grundy coloring, it is not maximum.

Thus, from the above cases, we have $\Gamma[S(L_n)]=\begin{cases}3, & n=2,3\\ 4, & n\geq 4\end{cases}$

Theorem 3.4: For $n \geq 2$, the grundy number for triangular ladder graph $T(L_n)$ is given by
$\Gamma[T(L_n)]=\begin{cases}n+1, & n=2,3\\ 5, & n\geq 4\end{cases}$

Proof: Consider a triangular ladder graph $T(L_n)$ with vertex set $V[T(L_n)]=\{u_i:1\leq i\leq n\}\cup\{v_j:1\leq j\leq n\}$ and edge set $E[T(L_n)]=\{u_iu_{i+1}:1\leq i<n\}\cup\{v_jv_{j+1}:1\leq j<n\}\cup\{u_iv_j:i,j\in[1,n]\,\&\,i=j\}\cup\{u_iv_{i+1}:1\leq i<n\}$ with $|V[T(L_n)]|=2n$ and $|E[T(L_n)]|=4n-3$.

$\therefore$ We have $\Delta[T(L_n)]=\begin{cases}n+1, & n=2,3\\ 4, & n\geq 4\end{cases}$ and $\delta[T(L_n)]=2$.

Consider the colors $C_1,C_2,C_3,\ldots$ and assign the colors as follows.

Case 1: When $n=2,3$

Let us consider the mapping $\lambda:V[T(L_n)]\rightarrow\{C_k:1\leq k\leq n+1\}$ and assign the colors as follows.

Subcase 1: For $n=2$

- $\lambda(v_j)=C_{j+1}\ \forall\ j\in[1,n]$
- $\lambda(u_i)=C_i\ \forall\ i\in[1,n]$

Thus, $\Gamma[T(L_n)]=3$. Suppose $\Gamma[T(L_n)]>3$, then the vertex u_2 colored with C_2 is not adjacent with C_1 for the mapping $\lambda(u_i)=\begin{cases}C_{i+2}, & i=1\\ C_i, & i=2\end{cases}$ and $\lambda(v_j)=\begin{cases}C_j, & j=1\\ C_{j+2}, & j=2\end{cases}$ which contradicts the definition of grundy coloring.

Subcase 2: When $n=3$

- $\lambda(u_i)=C_4$ & $\lambda(v_j)=C_3\ \forall\ i=j=2$.
- $\lambda(u_1)=\lambda(v_n)=C_2$
- $\lambda(v_1)=\lambda(u_n)=C_1$

Thus, $\Gamma[T(L_n)]=4$. Suppose $\Gamma[T(L_n)]>4$, then the vertex v_n colored with C_4 is not adjacent with C_2 for the mapping $\lambda(u_2)=C_5$, $\lambda(v_j)=C_{j+1}\ \forall\ j=2,3$, $\lambda(u_1)=C_2$ & $\lambda(u_n)=\lambda(v_1)=C_1$ which contradicts the definition of grundy coloring.

$\therefore$ From the above subcases, we have $\Gamma[T(L_n)]=n+1$ for $n=2,3$.

Case 2: When $n\geq 4$

Consider the mapping $\pi:V[T(L_n)]\rightarrow\{C_l:1\leq l\leq 4\}$ such that

- $\pi(v_1)=\pi(u_3)=C_1$ & $\pi(u_1)=\pi(v_4)=C_2$
- $\pi(u_2)=C_5$ & $\pi(u_4)=C_3$
- $\pi(v_j)=C_{j+1}\ \forall\ j=2,3$

- For $i, j \in [5,n]$, $\pi(u_i) = \begin{cases} C_2, & i \equiv 0 \bmod 2 \\ C_1, & i \equiv 1 \bmod 2 \end{cases}$ and $\pi(v_j) = \begin{cases} C_4, & j \equiv 1 \bmod 2 \\ C_3, & j \equiv 0 \bmod 2 \end{cases}$

Obviously, $\Gamma[T(L_n)] = 5$ for $n \geq 4$. Suppose $\Gamma[T(L_n)] < 5$, eventhough it satisfies the definition of grundy coloring, it is not maximum.

Thus, from the above cases, we have $\Gamma[T(L_n)] = \begin{cases} n+1, & n = 2,3 \\ 5, & n \geq 4 \end{cases}$

Theorem 3.5: For $n \geq 3$, the grundy number for open triangular ladder graph $OT(L_n)$ is given by $\Gamma[OT(L_n)] = \begin{cases} n, & n = 3,4 \\ 5, & n \geq 5 \end{cases}$

Proof: Consider a open triangular ladder graph $OT(L_n)$ with vertex set $V[OT(L_n)] = \{u_i : 1 \leq i \leq n\} \cup \{v_j : 1 \leq j \leq n\}$ and edge set $E[OT(L_n)] = \{u_i u_{i+1} : 1 \leq i < n\} \cup \{v_j v_{j+1} : 1 \leq j < n\} \cup \{u_i v_j : i, j \in (1,n) \,\&\, i = j\} \cup \{u_i v_{i+1} : 1 \leq i < n\}$ with $|V[OT(L_n)]| = 2n$ and $|E[OT(L_n)]| = 4n-5$.

$\therefore$ We have $\Delta[OT(L_n)] = 4$ and $\delta[OT(L_n)] = 1$.

Consider the colors $C_1, C_2, C_3, \ldots$ and assign the colors as follows.

Case 1: When $n = 3,4$

Let us consider the mapping $\mu : V[OT(L_n)] \to \{C_k : 1 \leq k \leq n\}$ and assign the colors as follows.

Subcase 1: For $n = 3$

- $\mu(u_1) = C_3$ & $\mu(v_1) = C_1$
- $\mu(u_i) = C_{i-1}$ $\forall$ $i \in [2,n]$
- $\mu(v_j) = C_j$ $\forall$ $j \in [2,n]$

Thus, $\Gamma[OT(L_n)] = 3$. Suppose $\Gamma[OT(L_n)] > 3$, then some vertex colored with C_k is not adjacent with all C_{k-1} colors which contradicts grundy coloring. For instance, $\Gamma[OT(L_n)] = 4$ then the vertex v_n colored with C_2 is not adjacent with C_1 for the mapping $\mu(u_1) = \mu(v_1) = \mu(u_n) = C_1$, $\mu(v_n) = C_2$, $\mu(v_{n-1}) = C_3$ & $\mu(u_{n-1}) = C_4$. This leads to contradiction.

Subcase 2: For $n = 4$

- $\mu(v_1) = \mu(v_n) = \mu(u_n) = C_1$ & $\mu(u_1) = C_2$
- $\mu(u_i) = C_{i-1}$ $\forall$ $i \in [2,n)$
- $\mu(v_j) = C_{j+1}$ $\forall$ $j \in [2,n)$

Thus, $\Gamma[OT(L_n)] = 4$. Suppose $\Gamma[OT(L_n)] > 4$, then the vertex v_n colored with C_2 is not adjacent with C_1 for the mapping $\mu(u_2) = C_5$, $\mu(v_{n-1}) = C_4$, $\mu(u_{n-1}) = C_3$, $\mu(u_1) = \mu(v_1) = \mu(v_n) = C_2$ & $\mu(u_n) = \mu(v_{n-2}) = C_1$ which contradicts the definition of grundy coloring.

Case 2: When $n \geq 5$

Consider the mapping $\psi : V[OT(L_n)] \to \{C_l : 1 \leq l \leq 5\}$ such that

- $\psi(v_j) = C_j$
- $\psi(u_1) = C_3$, $\psi(u_3) = C_5$ & $\psi(u_i) = C_{i/2}$ $\forall$ $i = 2,4$

- $\psi(u_n) = C_1$
- For $5 \le i \le n-1$, $\psi(u_i) = \begin{cases} C_4, & i \equiv 0 \bmod 2 \\ C_3, & i \equiv 1 \bmod 2 \end{cases}$
- For $5 \le j \le n$, $\psi(v_j) = \begin{cases} C_2, & j \equiv 0 \bmod 2 \\ C_1, & j \equiv 1 \bmod 2 \end{cases}$

Obviously, $\Gamma[OT(L_n)] = 5$ for $n \ge 5$. Suppose $\Gamma[OT(L_n)] < 5$, eventhough it satisfies the definition of grundy coloring, it is not maximum.

Thus, from the above cases, we have $\Gamma[OT(L_n)] = \begin{cases} n, & n = 3,4 \\ 5, & n \ge 5 \end{cases}$

Theorem 3.6: For $n \ge 3$, the grundy number for circular ladder graph $C(L_n)$ is given by $\Gamma[C(L_n)] = 4$

Proof: Consider a circular ladder graph $C(L_n)$ with vertex set $V[C(L_n)] = \{u_i : 1 \le i \le n\} \cup \{v_j : 1 \le j \le n\}$ and edge set $E[C(L_n)] = \{u_1 u_n\} \cup \{v_1 v_n\} \cup \{u_i u_{i+1} : 1 \le i < n\} \cup \{v_j v_{j+1} : 1 \le j < n\} \cup \{u_i v_j : i, j \in [1, n] \,\&\, i = j\}$ with $|V[C(L_n)]| = 2n$ and $|E[C(L_n)]| = 3n$.

$\therefore$ We have $\Delta[C(L_n)] = \delta[C(L_n)] = 3$.

Let us consider the mapping $\lambda : V[C(L_n)] \to \{C_k : 1 \le k \le 4\}$ and assign the colors as follows.

- For $i = j = 1$, $\lambda(u_i) = C_{i+1}$ & $\lambda(v_j) = C_j$
- For $i = j = 2$, $\lambda(u_i) = C_{2i}$ & $\lambda(v_j) = C_{j+1}$
- For $i, j \in [3, n)$, $\lambda(u_i) = \begin{cases} C_2, & i \equiv 0 \bmod 2 \\ C_1, & i \equiv 1 \bmod 2 \end{cases}$ and $\lambda(v_j) = \begin{cases} C_2, & j \equiv 1 \bmod 2 \\ C_1, & j \equiv 0 \bmod 2 \end{cases}$
- $\lambda(u_n) = \begin{cases} C_3, & n \equiv 0 \bmod 2 \\ C_1, & n \equiv 1 \bmod 2 \end{cases}$ and $\lambda(v_n) = \begin{cases} C_4, & n \equiv 0 \bmod 2 \\ C_2, & n \equiv 1 \bmod 2 \end{cases}$

Obviously, $\Gamma[C(L_n)] = 4$ for $n \ge 3$. Suppose $\Gamma[C(L_n)] < 4$, eventhough it satisfies the definition of grundy coloring, it is not maximum.

Thus, $\Gamma[C(L_n)] = 4$.

Theorem 3.7: For $n \ge 2$, the grundy number for mobius ladder graph M_n is given by

$$\Gamma(M_n) = \begin{cases} n-1, & n = 3 \\ 4, & n \ne 3 \end{cases}$$

Proof: Consider a mobius ladder graph M_n with vertex set $V(M_n) = \{u_i : 1 \le i \le n\} \cup \{v_j : 1 \le j \le n\}$ and edge set $E(M_n) = \{u_1 v_n\} \cup \{v_1 u_n\} \cup \{u_i u_{i+1} : 1 \le i < n\} \cup \{v_j v_{j+1} : 1 \le j < n\} \cup \{u_i v_j : i, j \in [1, n] \,\&\, i = j\}$ with $|V(M_n)| = 2n$ and $|E(M_n)| = 3n$.

$\therefore$ We have $\Delta(M_n) = \delta(M_n) = 3$.

Consider the colors $C_1, C_2, C_3, \cdots$ and assign the colors as follows.

Case 1: When $n=3$

Let us consider the mapping $\alpha : V(M_n) \to \{C_s : 1 \le s \le n-1\}$ and assign the colors as follows.

- $\alpha(u_1) = \alpha(u_n) = C_2$
- $\alpha(v_1) = \alpha(v_n) = C_1$
- For $i = j = 2$, $\alpha(u_i) = C_{i-1}$ & $\alpha(v_j) = C_j$

Thus, $\Gamma(M_n) = 2$. Suppose $\Gamma(M_n) > 2$, then the vertices u_1 & v_2 colored with C_2 are not adjacent with C_1 for the mapping $\alpha(v_1) = \alpha(v_n) = \alpha(u_{n-1}) = C_3$, $\alpha(u_1) = \alpha(v_{n-1}) = C_2$ and $\alpha(u_n) = C_1$ which contradicts the definition of grundy coloring.

Case 2: When $n \ne 3$

Consider the mapping $\beta : V(M_n) \to \{C_t : 1 \le t \le 4\}$ such that

- $\beta(u_i) = C_{2i}$ & $\beta(v_j) = C_{j+1}$ $\forall$ $i = j = 2$
- For n ≡ 1 mod 2, $\beta(u_n) = C_3$ & $\beta(v_n) = C_4$

and then the remaining u_i vertices are sequentially colored by C_2 & C_1 whereas v_j vertices are sequentially colored by C_1 & C_2.

Obviously, $\Gamma(M_n) = 4$ for $n \ne 3$. Suppose $\Gamma(M_n) < 4$, eventhough it satisfies the definition of grundy coloring, it is not maximum.

Thus, from the above cases, we have $\Gamma(M_n) = \begin{cases} n-1, & n=3 \\ 4, & n \ne 3 \end{cases}$

Theorem 3.8: For $n \ge 2$, the grundy number for diagonal ladder graph $D(L_n)$ is given by

$$\Gamma[D(L_n)] = \begin{cases} 4, & n = 2,3 \\ 6, & n \ge 4 \end{cases}$$

Proof: Consider a diagonal ladder graph $D(L_n)$ with vertex set $V[D(L_n)] = \{u_i : 1 \le i \le n\} \cup \{v_j : 1 \le j \le n\}$ and edge set

$$E[D(L_n)] = \begin{cases} \{u_i u_{i+1} : 1 \le i < n\} \cup \{v_j v_{j+1} : 1 \le j < n\} \cup \{u_i v_j : i, j \in [1,n] \,\&\, i = j\} \cup \\ \{u_i v_{i+1} : 1 \le i < n\} \cup \{v_j u_{j+1} : 1 \le j < n\} \end{cases}$$

with $|V[D(L_n)]| = 2n$ and $|E[D(L_n)]| = 5n-4$.

$\therefore$ We have $\Delta[D(L_n)] = \begin{cases} n+1, & n=2 \\ 5, & n \ge 3 \end{cases}$ and $\delta[D(L_n)] = 3$.

Consider the colors $C_1, C_2, C_3, \ldots$ and assign the colors as follows.

Case 1: For $n = 2,3$

Let us consider the mapping $\theta : V[D(L_n)] \to \{C_m : 1 \le m \le 4\}$ and assign the colors as follows.

- $\theta(u_i) = C_4$ & $\theta(v_j) = C_3$ $\forall$ $i = j = 2$

the remaining u_i vertices are colored by C_2 & v_j vertices are colored by C_1.

Thus, $\Gamma[D(L_n)] = 4$ for $n = 2,3$.

Obviously, $\Gamma[D(L_2)] = 4$.

In the case of $n=3$, Suppose $\Gamma[D(L_3)]>4$, then the vertex v_n colored with C_3 is not adjacent with C_2 for the mapping $\theta(u_2)=C_5$, $\theta(v_2)=C_4$, $\theta(v_n)=C_3$, $\theta(u_1)=C_2$ & $\theta(v_1)=\theta(u_n)=C_1$ which contradicts the definition of grundy coloring.

Case 2: For $n\geq 4$

Consider the mapping $\phi:V[D(L_n)]\to\{C_n:1\leq n\leq 6\}$ and assign the colors as follows.

- $\phi(u_i)=\begin{cases}C_4, & i\equiv 0\bmod 3\\ C_2, & i\equiv 1\bmod 3\\ C_6, & i\equiv 2\bmod 3\end{cases}$ and $\phi(v_j)=\begin{cases}C_3, & j\equiv 0\bmod 3\\ C_1, & j\equiv 1\bmod 3\\ C_5, & j\equiv 2\bmod 3\end{cases}$ such that $i,j\in[1,n]$ $\forall$ $n\equiv 1\bmod 3$, $i,j\in[1,n)$ $\forall$ $n\equiv 2\bmod 3$ & $i,j\in[1,n-1)$ $\forall$ $n\equiv 0\bmod 3$

and then the remaining u_i vertices are sequentially colored by C_3 & C_2 whereas v_j vertices are sequentially colored by C_4 & C_1.

Obviously, $\Gamma[D(L_n)]=6$ for $n\geq 4$.

Suppose $\Gamma[D(L_n)]<6$, eventhough it satisfies the definition of grundy coloring, it is not maximum.

Thus, from the above cases, we have $\Gamma[D(L_n)]=\begin{cases}4, & n=2,3\\ 6, & n\geq 4\end{cases}$

Theorem 3.9: For $n\geq 3$, the grundy number for open diagonal ladder graph $OD(L_n)$ is given by

$$\Gamma[OD(L_n)]=\begin{cases}n, & n=3\\ 5, & n=4,5\\ 6, & n\geq 6\end{cases}$$

Proof: Consider an open diagonal ladder graph $OD(L_n)$ with vertex set $V[OD(L_n)]=\{u_i:1\leq i\leq n\}\cup\{v_j:1\leq j\leq n\}$ and edge set

$$E[OD(L_n)]=\begin{Bmatrix}\{u_iu_{i+1}:1\leq i<n\}\cup\{v_jv_{j+1}:1\leq j<n\}\cup\{u_iv_j:i,j\in(1,n)\,\&\,i=j\}\\ \cup\{u_iv_{i+1}:1\leq i<n\}\cup\{v_ju_{j+1}:1\leq j<n\}\end{Bmatrix}$$

with $|V[OD(L_n)]|=2n$ and $|E[OD(L_n)]|=5n-6$.

$\therefore$ We have $\Delta[OD(L_n)]=5$ and $\delta[OD(L_n)]=2$.

Consider the colors $C_1,C_2,C_3,\ldots$ and assign the colors as follows.

Case 1: When $n=3$

Let us consider the mapping $\theta:V[OD(L_n)]\to\{C_p:1\leq p\leq 4\}$ such that

$\theta(u_i)=C_3$ & $\theta(v_j)=C_2$ $\forall$ $i=j=2$

$\theta(u_1)=\theta(u_n)=\theta(v_1)=\theta(v_n)=C_1$

Thus, $\Gamma[OD(L_n)]=3$ for $n=3$.

Suppose $\Gamma[OD(L_n)]>3$, then some vertex u_i or v_j colored with C_p is not adjacent with all C_{p-1} colors. For instance, $\Gamma[OD(L_n)]=4$ then the vertex u_n colored with C_2 is not adjacent with C_1 for the mapping $\theta(u_1)=\theta(v_1)=\theta(v_n)=C_1$, $\theta(u_n)=C_2$, $\theta(v_2)=C_3$ & $\theta(u_2)=C_4$ which contradicts the definition of grundy coloring.

Case 2: When $n=4,5$

Consider the mapping $\tau : V[OD(L_n)] \rightarrow \{C_q : 1 \le q \le 5\}$ such that

- $\tau(u_i) = C_i \ \forall \ i \in [1,3]$
- $\tau(u_4) = C_1$
- $\tau(v_j) = C_{j+2} \ \forall \ j = 2,3$

and then the remaining vertices are colored by greedy strategy.

Thus, $\Gamma[OD(L_n)] = 5$ for $n = 4,5$.

Suppose $\Gamma[OD(L_n)] > 5$, then some vertex u_i or v_j colored with C_q is not adjacent with all C_{q-1} colors. For instance, $\Gamma[OD(L_4)] = 6$ then the vertices u_1 & u_n colored with C_2 are not adjacent with C_1 for the mapping $\tau(v_1) = \tau(v_n) = C_1$, $\tau(u_1) = \tau(u_n) = C_2$ and $\tau(u_i) = C_{i+3}$ & $\tau(v_j) = C_{j+1} \ \forall \ i, j = 2,3$ which contradicts the definition of grundy coloring.

Case 3: When $n \ge 6$

Let us assume the mapping $\eta : V[OD(L_n)] \rightarrow \{C_r : 1 \le r \le 6\}$ and assign the colors as follows.

Subcase 1: For n ≡ 0 mod3

- $\eta(u_1) = \eta(v_1) = \eta(u_n) = \eta(v_n) = C_3$

Subcase 2: For n ≡ 1 mod3

- $\eta(v_{n-1}) = C_4$
- $\eta(u_1) = \eta(v_1) = \eta(u_{n-1}) = C_3$
- $\eta(u_n) = \eta(v_n) = C_1$

Subcase 3: For n ≡ 2 mod3

- $\eta(v_{n-2}) = C_4$
- $\eta(u_1) = \eta(v_1) = \eta(u_n) = \eta(v_n) = \eta(u_{n-2}) = C_3$
- $\eta(v_{n-1}) = C_2$
- $\eta(u_{n-1}) = C_1$

Then the remaining u_i & v_j vertices(where $i, j \in (1,n) \ \forall \ n \equiv 0 \bmod 3$, $i, j \in (1, n-1) \ \forall \ n \equiv 1 \bmod 3$ & $i, j \in (1, n-2) \ \forall \ n \equiv 2 \bmod 3$) are colored by $\eta(u_i) = \begin{cases} C_6, & i \equiv 0 \bmod 3 \\ C_4, & i \equiv 1 \bmod 3 \\ C_2, & i \equiv 2 \bmod 3 \end{cases}$ and

$$\eta(v_j) = \begin{cases} C_5, & j \equiv 0 \bmod 3 \\ C_3, & j \equiv 1 \bmod 3 \\ C_1, & j \equiv 2 \bmod 3 \end{cases}$$

Obviously, $\Gamma[OD(L_n)] = 6$ for $n \ge 6$.

Suppose $\Gamma[OD(L_n)] < 6$, eventhough it satisfies the definition of grundy coloring, it is not maximum.

Thus, from the above cases, we have

$$\Gamma[OD(L_n)] = \begin{cases} n, & n = 3 \\ 5, & n = 4,5 \\ 6, & n \ge 6 \end{cases}$$

REFERENCES

1. Brice Effantin, A note on Grundy coloring of central graphs, Australasian Journal of Combinatorics, Vol. 68(3), pp. 346-356, 2017.

2. Brice Effantin and Hamamache Kheddouci, Grundy number of graphs, Discussiones Mathematicae Graph Theory, Vol. 27, pp. 5–18, 2007.
3. Claude A. Christen and Stanley M. Selkow, Some Perfect Coloring Properties of Graphs, Journal of Combinatorial Theory, Series B, Vol. 27, pp. 49-59, 1979.
4. C. Jayasekaran and J. Little Flower, On Edge Trimagic Labeling of Umbrella, Dumb Bell and Circular Ladder Graphs, Annals of Pure and Applied Mathematics, Vol. 13(1), pp. 73-87, 2017.
5. Manouchehr Zaker, Results on the Grundy Chromatic Number of Graphs, Discrete Mathematics, Vol. 306, pp. 3166–3173, 2006.
6. Marie Asté, Frédéric Havet and Claudia Linhares-Sales, Grundy number and products of graphs, Discrete Mathematics, Vol. 310, pp. 1482–1490, 2010.
7. Nancy E. Clarke, Stephen Finbow, Shannon Fitzpatrick, Margaret-Ellen Messinger, Rebecca Milley and Richard J. Nowakowski, A Note on the Grundy Number and Graph Products, Discrete Applied Mathematics, Vol. 202, pp. 1-7, 2016.
8. Shaily Verma, B.S. Panda, Grundy Coloring in Some Subclasses of Bipartite Graphs and their Complements, Information Processing Letters, Vol. 163 (105999), 2020.
9. P. Sumathi, A. Rathi and A. Mahalakshmi, Quotient Labeling of Corona of Ladder Graphs, International Journal of Innovative Research in Applied Sciences and Engineering, Vol. 1(3), pp. 80-85, 2017.
10. P. Sumathi and A. Rathi, Quotient Labeling of Some Ladder Graphs, American Journal of Engineering Research, Vol. 7(12), pp. 38-42, 2018.
11. Zixing Tang, Baoyindureng Wu and Lin Hu, More Bounds for the Grundy Number of Graphs, Journal of Combinatorial Optimization, Vol. 33, pp. 580-589, 2017.